Master's Internship Report
Decidability of reachability for continuous linear time systems with constrained controls

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Summary

General Context

The problems of controllability and reachability in linear time invariant systems especially for bounded controls are some of the most fundamental questions asked in the control theory with widespread applications in circuit design, signal processing, cyberphysical systems and many other areas. It is therefore favourable to study and understand them. Proving decidability for these problems could potentially help in the verification of hybird systems as several safety properties can be represented as avoiding or not reaching certain bad states.

Problem

At first, we considered the system $x' = Ax + u(t)$ where $u(t)$ takes values in a convex polytope and is measurable, but very quickly we specialized into fixing that polytope to be a unit hypercube around origin. We wish to decide if a given point $\overrightarrow{y}$ could be reached from origin via some valid "control" u.
The basic setting of the problem (even the case of convex polytope) is considered by many authors(for eg. see [10],[5]) and several characterisations of the reachability space for several special cases are given. But none of the authors explicitly talked about decidability and it didn't seem obvious for some of those characterisations to be verifiable by a machine. So, considering those characterisations and trying to prove decidability for them seemed a fairly obvious next step.

Contribution

I initially studied the problem in low dimension [dimension 2] and described the reachability space in a more explicit manner which highlights the difficulty in decidability already in dimension 2. Then we considered some special cases and showed their decidability assuming decidability of certain first order theory (theory of reals with exponential function , reals with exponential,cosine and sine on bounded inputs) which is although currently open, widely believed to be decidable. We also proved time bounded reachability (whether you can reach in time $\leqslant T$ where T is part of the input) from the same assumption with a similar method. Then we tried to prove some hardness results, as the problem was seemingly hard. We were able to reduce another hard problem (continuous positivity) to a "slightly" generalised version (set reachability instead of point).

Arguments supporting its validity

One of the main drawbacks here is that we failed to prove any kind of hardness results for the concerning problem as that would have literally showed us why this problem is hard. Nevertheless, the proofs presented in the decidability section of the report all very much depend on the assumptions considered and it doesn't seem to be very obvious to generalise them.

Future work

Since the problem is apparently hard, we would like to have some hardness or undecidability results. To this end, we are considering some existing hard problems in number theory and try to reduce them to our problem. One of the main contributions I feel is to link the decidability of some other problem from a different area to the decidability of this problem. This would mean that any new techniques and proofs developed there could possibly result in some new developments here.
From here, as mentioned above, an obvious next good question would be to show some hardness result.

1 Introduction

The paper [4] proves some hardness results and decidability results in special cases (on the structure of the state matrix A) for reachability from origin in discrete linear systems with polytope control set. This internship started in spirit to work out the same,but with continuous time systems. Several books and papers talk about an equivalent problem (see lemma 2) (target is origin instead of initial vector, called the controllability problem) when the control set is a linear transformation of the hypercube $[-1,1]^m$. They give a sufficient condition for entire space to be controllable to 0 and also describe the structure of the controllabiltity set when the above condition is not met, but it doesn't seem obvious to come up with a decision procedure from the description. As far as we know, no author explicitly talks about the decidability or undecidability of the control problem in the general case.

Related Work. For a general continuous linear system $x' = Ax + Bu$ (let us call it by (S)), $u \in \Omega$, define null-controllability region C_{null} to be the set of points which can be steered to 0. We say that (S) is globally null controllable if $C_{null} = R^n$. Many different authors considered these problems (controllability and global null controllability) for different classes of sets Ω and sometimes even for time varying matrices.

Kalman ([7]) showed that when $\Omega = \mathbb{R}^m$, (S) is globally null controllable iff (A,B) is controllable (ref. definition 1). Lee and Markus([8]) considered Ω such that $0 \in \Omega \subset \mathbb{R}^m$ and showed that if (A,B) is controllable and all eigen values have negative real parts, then (S) is globally null-controllable. Sontag ([12]) considered the problem of asymptotic null-controllability which asks if there is a control which reaches the origin in the limit. Summers ([13]) discussed about over estimation of the reachable set (from origin) by n-dimensional ellipsoids when $\Omega = [-1,1]^m$. Schmitendorf ([11]) considers time varying matrices $A(t)$ and $B(t)$ and gives a characterisation for a given point to be controllable when Ω is compact.

Main result(s): Consider the system $x' = Ax + u$ $u \in \Omega \subset \mathbb{R}^n$. We define when a matrix A is called normal, and obtain a closed form expression for points on the boundary of the reachable set (from origin) for 2-dimensional systems whose state matrix is normal and when the control is $[-1,1]^2$. We also show decidability of the concerned problem for all dimensions with $\Omega = [-1,1]^n$ with the matrix A being supernormal (something which we will define later) , subject to the decidability of first-order theory of reals with exponential function. Further, we show the decidability of time bounded reachability and also in 2 dimensions with no restrictions on eigen values of A, subject to decidability of first order theory of reals with exponential function along with cosine and sine functions with bounded domain. Lastly, we show that the set reachability when considered with convex controls is continuous positivity hard.

Report organisation: In section 2 we briefly touch upon the techniques that we will use from linear algebra along with existing work on which we build upon. Then section 3 concerns with the decidability results and in section 4 we try to show Hardness of the problem.

2 Preliminaries and previous work

Jordan decomposition and matrix exponential

Given a square matrix A of order n with rational entries, one can find matrices P and Λ (possibly with complex algebraic entries) such that $A = P\Lambda P^{-1}$. Here Λ is a block diagonal matrix $\text{diag}(J_1, J_2, \ldots, J_m)$ where each J_i is a square matrix of order n_i and takes the following form.

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \ldots & 0 \\ 0 & \lambda_i & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_i & 1 \\ 0 & 0 & \ldots & 0 & \lambda_i \end{bmatrix}$$

where the λ_i would be one of the eigen values of A. Each such J_i is called a jordan block and it maybe that a single eigenvalue may have multiple jordan blocks of different sizes. Also for each block J_i the corresponding columns in P are the generalized eigen vectors of the eigenvalue in J_i.

One of the uses of jordan decomposition is to calculate exponential of a matrix. Given a matrix A, e^A is defined to be the limit of the series $\sum_{i=0}^{s} \frac{A^i}{i!}$ as $s \to \infty$. The limit exists for every matrix A and is also non-singular. From the above definition it is clear that if $A = P\Lambda P^{-1}$, then $e^{At} = Pe^{\Lambda t}P^{-1}$.

If $\Lambda = \text{diag}(J_1, J_2, \ldots, J_m)$, then it can be shown that $e^{\Lambda t} = \text{diag}(e^{J_1 t}, e^{J_2 t}, \ldots, e^{J_m t})$. One can obtain a closed form expression for a jordan block $e^{J_i t}$.

$$e^{J_i t} = e^{\lambda_i t} \begin{bmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{k-1}}{(k-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{k-2}}{(k-2)!} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & t \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix}$$

where k is the size of the jordan block and λ_i is the associated eigen value.
Consider two vectors c and b in $\mathbb{R}^n$. Define a function $f_{c,A,b}(t) = c^T e^{At} b$. Then it can be shown (ref. [1]) that $f_{c,A,b}(t) = \Sigma_{j=1}^m P_j(t) e^{\theta_j t}$ where each θ_j is an eigenvalue of A. Since we will be encountering the above function many times, it might be better to state some properties here which we will use later.

Lemma 1. *Let $f_{c,A,b}(t)$ be the function defined above. Then*

(1) If $f_{c,A,b} \not\equiv 0$, then the number of zeros of $f_{c,A,b}$ in any bounded interval is finite.

(2) for given A, b, $\forall c$ $f_{c,A,b} \not\equiv 0 \iff (A, b)$ is controllable.

The proof of above lemma can be found in any standard text (for ex. [5]).

Exisitng work

Definition 1 (Controllable)**.** A pair of matrices (A, B) is called controllable if $\mathrm{rank}([B, AB, \ldots, A^{n-1}B]) = n$ where A is an $n \times n$ matrix.

Definition 2 (Normality)**.** A pair of matrices (A, B) is called normal if (A, b_i) is controllable for every i. b_i is the i^{th} column of B.

We call matrix A to be normal if the pair (A, I_n) is normal

Definition 3 (Strict convexity)**.** A convex set S is strictly convex if the line segement joining any two distinct boundary points of S lies entirely inside S except for those two points *i.e.* $\forall z_1, z_2 \in \partial S$, $\alpha z_1 + (1-\alpha) z_2 \notin \partial S$, $\alpha \in (0, 1)$

Strict convexity can also be characterised interms of tangential hyperplanes. A convex set S is strictly convex $\iff$ $\forall c \neq 0\ \exists!\ z_{\max} \in \partial S\ c^T z_m ax = \max_{z \in \overline{S}} c^T z$
In this document, $\|x\|$ denotes the infinity norm of vectors $x \in \mathbb{C}^n$ and $\|A\|$ the induced norm on matrices $A \in \mathbb{C}^{n\times n}$. Recall that any induced norm is consistent ($\|Ax\| \leqslant \|A\|\|x\|$) and therefore submultiplicative ($\|AB\| \leqslant \|A\|\|B\|$). Given a matrix $A \in \mathbb{C}^{n\times n}$, e^A denotes the matrix exponential of A. In particular, we have that $\|e^A\| \leqslant e^{\|A\|}$.
Let $A \in \mathbb{R}^{n\times n}$ a matrix, $x_0 \in \mathbb{R}^n$ and $\mathbb{U} \subset \mathbb{R}^n$. Given $u : \mathbb{R} \to \mathbb{U}$ measurable, we consider the initial value problem

$$x(0) = x_0, \qquad x'(t) = Ax(t) + u(t). \tag{1}$$

The solution to the above system is given by

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-s)} u(s)\, ds.$$

Intuitively, x is the *state* of physical system and u is the *control* applied to it, therefore $\mathbb{U}$ encodes the set of available controls. We are interested in the following decision problems about reachability:

Problem (Point reachability)**.** *Given a square matrix $A \in \mathbb{Q}^{n\times n}$, a vector $y \in \mathbb{Q}^n$ and a bounded convex set $\mathbb{U} \subset \mathbb{R}^n$ effectively definable, decide if there exists $T \geqslant 0$ and $u : [0, T] \to \mathbb{U}$ measurable such that the solution x to* (1) *satisfies $x(0) = 0$ and $x(T) = y$.*

Problem (Set reachability)**.** *Given a square matrix $A \in \mathbb{Q}^{n\times n}$ and two sets $\mathbb{U}, \mathbb{T} \subset \mathbb{R}^n$ effectively definable, decide if there exists $T \geqslant 0$ and $u : [0, T] \to \mathbb{U}$ measurable such that the solution x to* (1) *satisfies $x(0) = 0$ and $x(T) \in \mathbb{T}$.*

We will often assume that the control set $\mathbb{U}$ is the hyper cube $[-1, 1]^n$, or a scaled hypercube $B[-1, 1]^m$ for some matrix B. Furthermore, following [6], define the matrix A to be semi-stable if all its eigenvalues have non-positive real parts, and as anti-stable if all of its eigenvalues have positive real-parts.

Observe that from the integral form of the solution to (1) null controllable region and the reachable region (from 0) of a general linear system (A, U) are given by

$$\mathcal{C} = \bigcup_{T\geq 0} \Big\{ -\int_0^T e^{-A\tau} u(\tau)\, d\tau \mid u \text{ is measurable and } u(\tau) \in U \Big\} \tag{2}$$

$$\mathcal{R} = \bigcup_{T\geq 0} \Big\{ \int_0^T e^{A\tau} u(\tau)\, d\tau \mid u \text{ is measurable and } u(\tau) \in U \Big\} \tag{3}$$

We reproduce the following theorems from [6]. For Theorem 1, it is assumed that the control set is $B[-1, 1]^m$ for some matrix B of order $n \times m$

Theorem 1. *Let (A, B) be controllable. Then,*

- *If A is semi-stable, then $\mathcal{C} = \mathbb{R}^n$*
- *If A is anti-stable, then $\mathcal{C}$ is a bounded convex open set containing the origin*
- *If $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ where $A_1 \in \mathbb{R}^{n_1 \times n_1}$ anti-stable and $A_2 \in \mathbb{R}^{n_2 \times n_2}$ semi-stable and B is partitioned as $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ accordingly, then $\mathcal{C} = \mathcal{C}_1 \times \mathbb{R}^{n_2}$ where $\mathcal{C}_1$ is the null controllable region for the system $\boldsymbol{X_1' = A_1 X_1(t) + B_1 u(t)}$*

In our case, when considering $\mathbb{U}$ to be $[-1, 1]^n$, we can apply Theorem 1 with $B = I_n$ the identity matrix. Then the condition that (A, B) be controllable is always satisfied and we can readily apply the theorem in this case.
The following lemma connects the null controllable region to our problem

Lemma 2. *The reachable region of the linear system given by (A, U) is same as the null controllable region for the system given by $(-A, -U)$*

The proof of the lemma is simple and can be seen directly from the formulas (2) and (3)

Theorem 2. *For the system $X' = AX + bv(t)$, when A is anti-stable and b is a vector in $\mathbb{R}^n$ such that (A, b) is controllable and $|v(t)| \leq 1 \forall t \geq 0$ the boundary of the null-controllable region is given by*

$$\partial\mathcal{C} = \left\{ \int_{-\infty}^{0} e^{A\tau} b \,\mathrm{sgn}(c^T e^{A\tau} b) \, d\tau \;\middle|\; c \in \mathbb{R}^n \setminus \{\mathbf{0}\} \right\}$$

and $\overline{\mathcal{C}}$ is strictly convex.

Since we will be using this theorem along with above lemma several times, we would like to reproduce the proof from [6].

Proof. Firstly, denote $\mathcal{C}(T)$ to be the set of points which can be steered to 0 in time T. Since v can take the value 0 $\implies$ $\mathcal{C}(T) \subseteq \mathcal{C}(T')\ \forall T < T'$.
$\implies \bigcup_{t \leqslant T} \mathcal{C}(t) = \mathcal{C}(T) = \left\{ -\int_0^T e^{-A\tau} bv(\tau)\, d\tau \mid v \text{ is measurable and } v(\tau) \in [-1, 1] \right\}$ and therefore since A is anti-stable we have $\overline{\mathcal{C}} = \left\{ -\int_0^\infty e^{-A\tau} bv(\tau)\, d\tau \mid v \text{ is measurable and } v(\tau) \in [-1, 1] \right\} = \left\{ \int_{-\infty}^0 e^{A\tau} bv(\tau)\, d\tau \mid v \text{ is measurable and } v(\tau) \in [-1, 1] \right\}$
One can see that $\overline{\mathcal{C}}$ is indeed convex and Theorem 1 guarantees it to be bounded.
Let $z^* \in \partial\mathcal{C}$. Then, there exists a non-zero vector $c \in \mathbb{R}^n$ such that

$$c^T z^* = \max_{z \in \overline{\mathcal{C}}} c^T z = \max_{v \text{ measurable}} \int_{-\infty}^{0} c^T e^{At} bv(t)\, dt$$

clearly $v_{\max}(t) = \mathrm{sgn}(c^T e^{At} b)$ maximizes R.H.S and since (A, b) is controllable, this would imply that $c^T e^{At} b \not\equiv 0$ and hence $v_{\max}(t)$ is piecewise constant for any non-zero c. consequently $v_{\max}$ is measurable, hence a valid control. If we show that this is the only control which maximizes the right hand side, then we would have shown that $\overline{\mathcal{C}}$ is strictly convex and as well as obtain the formula for the point on the boundary.
Consider any other control $v(t)$ different from $v_{\max}$. $\implies$ there exists a set $E_1 \subset (-\infty, 0]$ with a non-zero measure i.e. $\lambda(E_1) = \delta_1 > 0$ and $\epsilon_1 > 0$ such that $|v_{\max}(t) - v(t)| \geqslant \epsilon_1$ where $\lambda(.)$ denotes the lebesgue measure of a set. Further since the function $c^T e^{At} b$ is analytic in t, it vanishes on a set of measure zero. therefore $\exists\ E \subset E_1$ such that $\lambda(E) = \delta$ and $|c^T e^{At} b| \geqslant \epsilon$ for some $\epsilon > 0$.
Also observe that $c^T e^{At} bv_{\max}(t) = |c^T e^{At} b| \geqslant c^T e^{At} bv(t)\ \forall t$ and functions v.

$$\implies \int_{-\infty}^{0} c^T e^{At} b(v_{\max}(t) - v(t))\, dt$$

$$\geqslant \int_E c^T e^{At} b(v_{\max}(t) - v(t))\, dt = \int_E |c^T e^{At} b||(v_{\max}(t) - v(t))|\, dt \geqslant \delta\epsilon\epsilon_1 > 0$$

$$\implies \text{there is a unique point which maximizes } c^T z \text{ and it is given by } z^* = \int_{-\infty}^{0} e^{At} b \,\mathrm{sgn}(c^T e^{At} b)\, dt$$

We showed that any point on the boundary takes the above form.To show that any point which takes the above form is on the boundary is trivial.
This completes the proof. □

3 Special cases

In this section, unless otherwise mentioned we will assume that the matrix A is normal along with whatever special conditions being imposed on it and the $\mathbb{U}$ to be $[-1,1]^n$.

3.1 Diagonal matrices

Let $A = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. w.l.o.g assume $\lambda_i \geq \lambda_{i+1} \forall i$. Let k be the smallest number for which $\lambda_{k+1} < 0$. Then from theorem 1,theorem 2,lemma 2 and the fact that eigenvalues of $-A$ are negative of the eigenvalues of A we have that reachable region is given by $\mathcal{R} = \mathbb{R}^k \times \mathcal{R}_1$.

$$\mathcal{R}_1 = \bigcup_{T\geq 0} \left\{ \int_0^T \begin{bmatrix} e^{\lambda_{k+1}} u_{k+1}(t) \\ \vdots \\ e^{\lambda_n} u_n(t) \end{bmatrix} d\tau \mid u \text{ is measurable and } u(\tau) \in [-1,1]^{n-k} \right\}$$

The assumption that the control space is a hyper cube allows us to deal with each co-ordinate separately. This implies $\mathcal{R}_1 = \prod_{i=k+1}^n R_i$ where each R_i is given by $\bigcup_{T\geq 0} \left\{ \int_0^T e^{\lambda_i \tau} u(\tau) d\tau \mid u \text{ is measurable and } u(\tau) \in [-1,1] \right\}$.
Furthermore each R_i is a convex open symmetric set in $\mathbb{R}$ from theorem 1 and the fact that the control is symmetric. *i.e.* R_i is of the form $(-a_i, a_i)$ and this a_i is given by

$$\int_0^\infty e^{\lambda_i \tau} d\tau$$

from theorem 2 which equals $\frac{-1}{\lambda_i}$. Therefore deciding reachability in this case is trivial.

3.2 Diagonalizable matrices in 2 dimensions

In this subsection we will deal with 2×2 matrices which are diagonalizable over reals with distinct eigen values, *i.e.* $\exists$ invertible matrix P such that

$$A = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1}$$

w.l.o.g let $\lambda_1 < \lambda_2$. We wish to solve point reachability question for these kinds of matrices *i.e.* given a vector $y = (y_1, y_2)^t$ we wish to decide if it is possible to reach y starting from $\mathbf{0}$. A point y is reachable in a $(P\Lambda P^{-1}, U)$ system $\iff$ $P^{-1}y$ is reachable in $(\Lambda, P^{-1}U)$ system.
From now, we focus on the $(\Lambda, P^{-1}U)$ system. Depending on the signs of λ_i's we have three cases.

- $\lambda_1 \geq 0$: In this case the any vector is reachable as the reachability space is whole of $\mathbb{R}^2$
- $\lambda_1 < 0$ and $\lambda_2 \geq 0$: From theorem 1 , the reachability space is $(-a, a) \times \mathbb{R}$ where $a = \frac{(|(P^{-1})_{11}| + |(P^{-1})_{12}|)}{|\lambda_1|}$
- $\lambda_2 < 0$: In this case, from theorem 1 we see that the reachability set is convex and bounded in $\mathbb{R}^2$ containing the origin. we now focus on this particular case.

we have $\Lambda = \begin{bmatrix} -r_1 & 0 \\ 0 & -r_2 \end{bmatrix}$ and let $P^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $r_1 > 0$, $r_2 > 0$ and $r_1 > r_2$
The reachability set $\mathcal{R}$ can be seen as $R_1 \oplus R_2$ where
R_1 is the reachability set for the $(\Lambda, [-1,1] * \begin{bmatrix} a \\ c \end{bmatrix})$ system and R_2 is the reachability set for the $(\Lambda, [-1,1] * \begin{bmatrix} b \\ d \end{bmatrix})$ system and $\oplus$ denotes minkowski sum operator.
Again from theorem 2 and from the fact that A is normal, we know that R_1 is open,convex and bounded and its boundary is given by

$$\partial R_1 = \left\{ \int_0^\infty e^{\Lambda y} \begin{bmatrix} a \\ c \end{bmatrix} \text{sgn}\left(\tau^T e^{\Lambda y} \begin{bmatrix} a \\ c \end{bmatrix} \right) dy \;\middle|\; \tau = (\cos\theta, \sin\theta)^T, \theta \in [0, 2\pi) \right\}$$

$$= \left\{ \int_0^\infty \begin{bmatrix} e^{-r_1 y} a \\ e^{-r_2 y} c \end{bmatrix} sgn(e^{-r_1 y} a \cos\theta + e^{-r_2 y} c \sin\theta) \; dy \;\middle|\; \theta \in [0, 2\pi) \right\}$$

$$= \left\{ \int_0^\infty \begin{bmatrix} e^{-r_1 y} a \\ e^{-r_2 y} c \end{bmatrix} sgn(e^{(r_2 - r_1) y} a \cos\theta + c \sin\theta) \; dy \;\middle|\; \theta \in [0, 2\pi) \right\}$$

We will assume $a \neq 0$ and $c \neq 0$ (similarly for b and d) as this is necessary for A to be normal. Depending on the value of θ we will have the following cases.

Case-I : $a\cos\theta\ c\sin\theta \geq 0$ In this case, the sgn function is constant either 1 or -1 depending on the sign of $a\cos\theta$ so the integral evaluates to $\pm(\frac{a}{r_1}, \frac{c}{r_2})$

Case-II : $a\cos\theta\ c\sin\theta < 0$ and $|a\cos\theta| < |c\sin\theta|$ even in this case the sgn function is constant, so the integral again evaluates to $\pm(\frac{a}{r_1}, \frac{c}{r_2})$

Case-III : $a\cos\theta\ \ c\sin\theta\ <\ 0$ and $|a\cos\theta| \geq |c\sin\theta|$ In this case, the function changes its sign at the point $t = 1/(r_2 - r_1) * \log(|(c/a)\tan\theta|)$. Assume initially the value is -1 until time t and then it changes to 1. The other case is just negative of this.

$$\int_0^\infty \begin{bmatrix} e^{-r_1 y} a \\ e^{-r_2 y} c \end{bmatrix} sgn(e^{(r_2 - r_1)y} a\cos\theta + c\sin\theta)\ dy = -\int_0^t \begin{bmatrix} e^{-r_1 y} a \\ e^{-r_2 y} c \end{bmatrix}\ dy + \int_t^\infty \begin{bmatrix} e^{-r_1 y} a \\ e^{-r_2 y} c \end{bmatrix}\ dy$$
$$= \begin{bmatrix} a/r_1 * e^{(-r_1 y)} \\ c/r_2 * e^{(-r_2 y)} \end{bmatrix}\Bigg|_0^t + \begin{bmatrix} a/r_1 * e^{(-r_1 y)} \\ c/r_2 * e^{(-r_2 y)} \end{bmatrix}\Bigg|_\infty^t$$
$$= \begin{bmatrix} a/r_1 * (2\alpha^{r_1} - 1) \\ c/r_2 * (2\alpha^{r_2} - 1) \end{bmatrix} \text{ where } \alpha = e^{-t} = (\tfrac{|c\tan\theta|}{|a|})^{\frac{1}{r_1 - r_2}}.$$

Observe that in this case we have that $\frac{|c\tan\theta|}{|a|} \leq 1$ and it can take every value between 0 and 1 depending on the value of θ.Therefore the boundary of the reachability set R_1 can be given by

$$\partial R_1 = \left\{ \pm \begin{bmatrix} a/r_1 * (2\alpha^{r_1} - 1) \\ c/r_2 * (2\alpha^{r_2} - 1) \end{bmatrix} \;\middle|\; \alpha = x^{\frac{1}{r_1 - r_2}}\ ,\ 0 \leq x \leq 1 \right\}.$$

Similarly we can describe the boundary for R_2. The graphs of the boundaries for an example matrix are shown below.

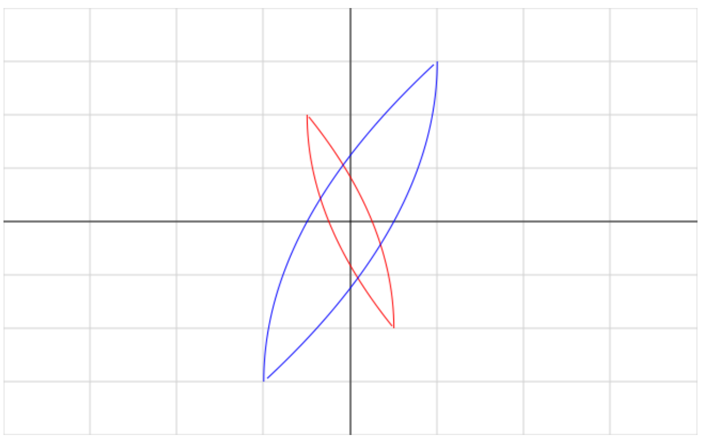

Figure 1: Boundaries of R_1 and R_2 for $r_1 = 2$, $r_2 = 1$, $a = 1$, $c = -2$, $b = 2$, $d = 3$

Now that we have got a reasonable description for the boundary of the reachability, we will try to give a little reason as to why even now, the reachability question remains hard even in the 2×2 case. Assume the vectors $[a, c]^T$ and $[b, d]^T$ are in the same quadrant of $\mathbb{R}^2$. Consider an angle θ such that we fall in the case-III for both those vectors. Note that it is possible since we assumed both vectors lie in the same quadrant. Now the boundary point corresponding to this angle would be

$$\begin{bmatrix} \frac{a}{r_1}\left(2\frac{|c\tan\theta|}{|a|}^{\frac{r_1}{r_1 - r_2}} - 1\right) + \frac{b}{r_1}\left(2\frac{|d\tan\theta|}{|b|}^{\frac{r_1}{r_1 - r_2}} - 1\right) \\ \frac{c}{r_2}\left(2\frac{|c\tan\theta|}{|a|}^{\frac{r_2}{r_1 - r_2}} - 1\right) + \frac{d}{r_2}\left(2\frac{|d\tan\theta|}{|b|}^{\frac{r_2}{r_1 - r_2}} - 1\right) \end{bmatrix}$$

Now one way to check for reachability given an algebraic point, would be go along that vector and see if that vector cuts the boundary or not. This would mean, for a decidability procedure we should be able to determine whether the ρ satisfying the below simultaneous equations is < 1 or not.

$$\begin{bmatrix} \frac{a}{r_1}\left(2\frac{|c\tan\theta|}{|a|}^{\frac{r_1}{r_1 - r_2}} - 1\right) + \frac{b}{r_1}\left(2\frac{|d\tan\theta|}{|b|}^{\frac{r_1}{r_1 - r_2}} - 1\right) \\ \frac{c}{r_2}\left(2\frac{|c\tan\theta|}{|a|}^{\frac{r_2}{r_1 - r_2}} - 1\right) + \frac{d}{r_2}\left(2\frac{|d\tan\theta|}{|b|}^{\frac{r_2}{r_1 - r_2}} - 1\right) \end{bmatrix} = \rho \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

Let $s = |\tan\theta|^{\frac{r_1}{r_1-r_2}}$, then dividing two co-ordinates and cross-multiplying, one can see that solving for $\tan\theta$ and thereby for ρ would involve numbers of the form $\kappa_1^{\kappa_2} + \kappa_3^{\kappa_4}$ where all κ_i are algebraic. And right now, very little is known about equations involving these kind of numbers.

3.3 2 dimensional matrices with complex eigen values

In this subsection, we determine the boundary of the reachability set of a 2 dimensional real matrix A with complex eigen values. Once again, we will assume that the real parts of these eigen values are negative as otherwise the entire space is reachable.

Since the matrix A is real, the eigen values will be conjugate to each other. Let the eigen values be $\lambda \pm i\theta$ with $\lambda < 0$ and $\theta > 0$. From jordan decomposition of A, we know that there exists a matrix $P \in GL_2(\mathbb{C})$ such that $A = P\begin{bmatrix}\lambda + i\theta & 0 \\ 0 & \lambda - i\theta\end{bmatrix}P^{-1}$. Also the columns of P are the eigen vectors of corresponding eigen values.

Claim 1. *Let A be the above matrix and Λ be the diagonal matrix with eigenvalues of A on its diagonal. Then $\exists$ real invertible matrix S and a real matrix Λ' such that* $\exp(At) = S\exp(\Lambda' t)S^{-1}$

Proof. we have $\Lambda = \begin{bmatrix}\lambda + i\theta & 0 \\ 0 & \lambda - i\theta\end{bmatrix}$. Since the matrix A is real, if $\mathbf{R_1} + \mathbf{iI_1}$ is an eigen vector for $\lambda + i\theta$, then $\mathbf{R_1} - \mathbf{iI_1}$ will be the eigen vector for $\lambda - i\theta$. This implies that $P = \begin{bmatrix}\mathbf{R_1} + \mathbf{iI_1} & \mathbf{R_1} - \mathbf{iI_1}\end{bmatrix}$.

consider $\Lambda' = \begin{bmatrix}\lambda & \theta \\ -\theta & \lambda\end{bmatrix} = \lambda I + \underbrace{\begin{bmatrix}0 & \theta \\ -\theta & 0\end{bmatrix}}_{\text{call this matrix B}}$. Then we have $e^{\Lambda' t} = e^{(\lambda I + B)t} = e^{\lambda t I + Bt} = e^{\lambda t}e^{Bt}$

It can be shown by induction that

$$(Bt)^{2k} = \begin{bmatrix}(-1)^k(\theta t)^{2k} & 0 \\ 0 & (-1)^k(\theta t)^{2k}\end{bmatrix} \text{ and } (Bt)^{2k+1} = \begin{bmatrix}0 & (-1)^k(\theta t)^{2k+1} \\ (-1)^{k+1}(\theta t)^{2k+1} & 0\end{bmatrix}$$

which implies $e^{\Lambda' t} = e^{\lambda t}\begin{bmatrix}\cos(\theta t) & \sin(\theta t) \\ -\sin(\theta t) & \cos(\theta t)\end{bmatrix}$

Next consider $S = P\begin{bmatrix}1 & 1 \\ i & -i\end{bmatrix}^{-1}$. Observe that the matrix $S = \begin{bmatrix}\mathbf{R_1} & \mathbf{I_1}\end{bmatrix}$ is real and also invertible by construction as P is invertible.

To prove $e^{At} = Pe^{\Lambda t}P^{-1} = Se^{\Lambda' t}S^{-1}$ is equivalent to proving $(S^{-1}P)e^{\Lambda t}(S^{-1}P)^{-1} = e^{\Lambda' t}$

Observe that $S^{-1}P = \begin{bmatrix}1 & 1 \\ i & -i\end{bmatrix}$ and it's inverse is $\frac{1}{2}\begin{bmatrix}1 & -i \\ 1 & i\end{bmatrix}$

$$\begin{aligned}
\implies (S^{-1}P)e^{\Lambda t}(S^{-1}P)^{-1} &= \begin{bmatrix}1 & 1 \\ i & -i\end{bmatrix} e^{\lambda t}\begin{bmatrix}e^{i\theta t} & 0 \\ 0 & e^{-i\theta t}\end{bmatrix}\frac{1}{2}\begin{bmatrix}1 & -i \\ 1 & i\end{bmatrix} \\
&= \frac{e^{\lambda t}}{2}\begin{bmatrix}e^{i\theta t} & e^{-i\theta t} \\ ie^{i\theta t} & -ie^{-i\theta t}\end{bmatrix}\begin{bmatrix}1 & -i \\ 1 & i\end{bmatrix} \\
&= e^{\lambda t}\begin{bmatrix}\cos\theta t & \sin\theta t \\ -\sin\theta t & \cos\theta t\end{bmatrix} \\
&= e^{\Lambda' t}
\end{aligned}$$

Also, Observe that $A = S\Lambda' S^{-1}$ □

Let $S^{-1} = \begin{bmatrix}a & b \\ c & d\end{bmatrix}$. Using the same trick as before, we focus on the $(\Lambda', S^{-1}\mathbb{U})$ system.

Since we have $\lambda < 0$, we can use Theorem 2 on each column of S^{-1} and using strong convexity we obtain the boundary of the reachability set to be

$$\left\{\sum_{i=1}^{2}\int_0^\infty e^{\Lambda' s}S_i^{-1}.\operatorname{sgn}(f^T e^{\Lambda' s}S_i^{-1})\,ds \mid f = (\cos\phi, \sin\phi)^T, \phi \in [0, 2\pi)\right\}$$

As both the integrals are similar in nature, we concentrate on a single integral and try to simplify it. We have

$$I_\phi = \int_0^\infty e^{\lambda s}\begin{bmatrix}\cos\theta s & \sin\theta s \\ -\sin\theta s & \cos\theta s\end{bmatrix}\begin{bmatrix}a \\ c\end{bmatrix}\operatorname{sgn}(f^T e^{\Lambda' s}S_1^{-1})\,ds$$

$$\begin{bmatrix}\cos\theta s & \sin\theta s\\ -\sin\theta s & \cos\theta s\end{bmatrix}\begin{bmatrix}a\\ c\end{bmatrix} = \sqrt{a^2+c^2}\begin{bmatrix}\sin(\theta s+\beta)\\ \cos(\theta s+\beta)\end{bmatrix} \quad \text{where } \sin\beta = \frac{a}{\sqrt{a^2+c^2}} \quad \text{and } \cos\beta = \frac{c}{\sqrt{a^2+c^2}}$$

$$\operatorname{sgn}(f^T e^{\Lambda' s} S_1^{-1}) = \operatorname{sgn}\left([\cos\phi \quad \sin\phi]^T \begin{bmatrix}\sin(\theta s+\beta)\\ \cos(\theta s+\beta)\end{bmatrix}\right) = \operatorname{sgn}(\sin(\theta s+\beta+\phi))$$

$$\implies I_\phi = \sqrt{a^2+c^2}\int_0^\infty e^{\lambda s}\begin{bmatrix}\sin(\theta s+\beta)\\ \cos(\theta s+\beta)\end{bmatrix}\operatorname{sgn}(\sin(\theta s+\beta+\phi))\, ds$$

viewing the above vector as a complex number with the 2^{nd} co-ordinate representing the real part, we get

$$I_\phi = \sqrt{a^2+c^2}e^{i\beta}\int_0^\infty e^{(\lambda+i\theta)s}\operatorname{sgn}(\sin(\theta s+\beta+\phi))\, ds$$

Let $t_{\beta,\phi}$ be the smallest $s > 0$ such that $\sin(\theta s+\beta+\phi) = 0$. Denote by $t^i_{\beta,\phi} = t_{\beta,\phi} + \frac{i\pi}{\theta} \forall\ i \geqslant 0$. Then

$$\begin{aligned}
I_\phi &= (c+ia)\operatorname{sgn}^+(\sin(\phi+\beta))\left(\int_0^{t^0_{\beta,\phi}} e^{(\lambda+i\theta)s}\, ds - \int_{t^0_{\beta,\phi}}^{t^1_{\beta,\phi}} e^{(\lambda+i\theta)s}\, ds + \int_{t^1_{\beta,\phi}}^{t^2_{\beta,\phi}} e^{(\lambda+i\theta)s}\, ds \ \ldots\right)\\
&= (c+ia)\operatorname{sgn}^+(\sin(\phi+\beta))\left(\int_0^{t^0_{\beta,\phi}} e^{(\lambda+i\theta)s}\, ds - \sum_{j=0}^{\infty}\int_{t^{2j}_{\beta,\phi}}^{t^{2j+1}_{\beta,\phi}} e^{(\lambda+i\theta)s}\, ds - \int_{t^{2j+1}_{\beta,\phi}}^{t^{2j+2}_{\beta,\phi}} e^{(\lambda+i\theta)s}\, ds\right)\\
&= \frac{c+ia}{\lambda+i\theta}\operatorname{sgn}^+(\sin(\phi+\beta))\left(e^{(\lambda+i\theta)t_{\beta,\phi}} - 1 - \sum_{j=0}^{\infty} e^{(\lambda+i\theta)t^{2j}_{\beta,\phi}}(2e^{(\lambda+i\theta)\frac{\pi}{\theta}} - 1 - e^{(\lambda+i\theta)\frac{2\pi}{\theta}})\right)\\
&= \frac{c+ia}{\lambda+i\theta}\operatorname{sgn}^+(\sin(\phi+\beta))\left(e^{(\lambda+i\theta)t_{\beta,\phi}} - 1 + e^{(\lambda+i\theta)t_{\beta,\phi}}\sum_{j=0}^{\infty}\left(e^{(\lambda+i\theta)}\right)^{2j\pi/\theta}\left(1+e^{\frac{2\pi\lambda}{\theta}} + 2e^{\frac{\pi\lambda}{\theta}}\right)\right)\\
&= \frac{c+ia}{\lambda+i\theta}\operatorname{sgn}^+(\sin(\phi+\beta))\left(e^{(\lambda+i\theta)t_{\beta,\phi}} - 1 + e^{(\lambda+i\theta)t_{\beta,\phi}}\left(1+e^{\frac{\lambda\pi}{\theta}}\right)^2\sum_{j=0}^{\infty}\left(e^{\frac{2\pi\lambda}{\theta}}\right)^j\right)\\
&= \frac{c+ia}{\lambda+i\theta}\operatorname{sgn}^+(\sin(\phi+\beta))\left(e^{(\lambda+i\theta)t_{\beta,\phi}}\left(1+\frac{1+e^{\frac{\lambda\pi}{\theta}}}{1-e^{\frac{\lambda\pi}{\theta}}}\right) - 1\right)\\
&= \frac{(c+ia)\operatorname{sgn}^+(\sin(\phi+\beta))}{(\lambda+i\theta)(1-e^{\frac{\lambda\pi}{\theta}})}\left(2e^{(\lambda+i\theta)t_{\beta,\phi}} + e^{\frac{\lambda\pi}{\theta}} - 1\right)\\
&= \frac{(c+ia).(\lambda-i\theta)\operatorname{sgn}^+(\sin(\phi+\beta))}{(\lambda^2+\theta^2)(1-e^{\frac{\lambda\pi}{\theta}})}\left(2e^{(\lambda+i\theta)t_{\beta,\phi}} + e^{\frac{\lambda\pi}{\theta}} - 1\right)\\
&= \frac{(c\lambda+a\theta+i(a\lambda-c\theta)).(2e^{(\lambda+i\theta)t_{\beta,\phi}})\operatorname{sgn}^+(\sin(\phi+\beta))}{(\lambda^2+\theta^2)(1-e^{\frac{\lambda\pi}{\theta}})} - \frac{(c\lambda+a\theta+i(a\lambda-c\theta))\operatorname{sgn}^+(\sin(\phi+\beta))}{(\lambda^2+\theta^2)}
\end{aligned}$$

where $\operatorname{sgn}^+(\sin(\phi+\beta))$ denotes $\lim_{s\to 0^+}\operatorname{sgn}(\sin(\theta s+\phi+\beta))$
For the second integral, everything is same except for the values of β and $t_{\beta,\phi}$ which were dependent on a, c change. $\implies$ the boundary points look like

$$\frac{2.\left[(\kappa_1+i\kappa_2)e^{(\lambda+i\theta)t_1} + (\kappa_3+i\kappa_4)e^{(\lambda+i\theta)t_2}\right]}{(\lambda^2+\theta^2)(1-e^{\frac{\lambda\pi}{\theta}})} - \frac{\kappa_1+\kappa_3+i(\kappa_2+\kappa_4)}{\lambda^2+\theta^2}$$

where all the κ_i's are algebraic and all of them along with t_1 and t_2 are dependent on ϕ and the matrix P^{-1}.
The aim of this subsection was to obtain a reasonably closed form expression for the points on the boundary which will be used later.

3.4 Existential theory of reals with exponential function ($\mathbb{R}_{\exp}$)

We introduce the theory of reals with exponential function in this subsection. Firstly, consider the first order formulae formed from the symbols $\mathcal{L} = \langle +, ., -, <, =, 0, 1\rangle$. When one considers the theory of the ordered fields along with an axiom stating that every positive number has a square root and an axiom schema stating that all polynomials of odd degree have at least one root, one obtains what is called the theory of real closed fields. Note that $\mathbb{R}$ is model for the above theory as it satisfies all the axioms. It was shown by Tarski in [14] that it is decidable to check if a sentence in the above theory is true or not by means of quantifier elimination which has non elementary complexity. Later it was proven in [2] that when one only considers the sentences of the form $\exists x_1,\, x_2,\, \ldots,\, x_n\ \Psi(x_1, x_2, \ldots, x_n)$ *i.e.* sentences where the quantifier is only $\exists$, one can decide the truth value in PSPACE. It is useful to note that already in this existential theory we can represent algebraic numbers *i.e.* for every $\alpha \in \mathbb{A}$, we have a formula $\psi_\alpha(y)$ such that $\psi_\alpha(y) \iff y = \alpha$.(For instance $\sqrt{2}$ can be represented as $y.y = (1+1) \wedge 0 < y$)

In the same [14] paper, Tarski asked whether there is procedure for deciding sentences in $\mathcal{L}_{\exp} = \langle +, ., -, <, =, 0, 1, \exp \rangle$ where exp is a unary function when considered over $\mathbb{R}$ with $\exp(x)$ denoting the exponential function e^x. This question was answered positively in [9] subject to Real Schanuel Conjecture. Note that in this theory ($\mathbb{R}_{\exp}$) we can represent logarithms of positive algebraic numbers and as a consequence, numbers of the form $\alpha_1^{\alpha_2}$ where both α_1 and $\alpha_2 \in \mathbb{A}$ and $\alpha_1 > 0$. For expressing $\log \alpha$ we use $\phi_\alpha(z) = \exists y\ \psi_\alpha(y) \wedge \exp z = y$. For expressing $\alpha_1^{\alpha_2}$ we rewrite it as $\exp(\alpha_2. \log \alpha_1)$ and use $\psi_{\alpha_1,\alpha_2}(y) = \exists y_1, y_2\ \psi_{\alpha_2}(y_1) \wedge \phi_{\alpha_1}(y_2) \wedge y = \exp(y_1.y_2)$

Also, in the same paper ([9]), Macintyre and Wilkie showed that for each natural number n,the first order theory of reals with $\exp, \cos_{\restriction [0,n]}, \sin_{\restriction [0,n]}$ is decidable assuming Schanuel's conjecture.

3.5 Reachability for matrices with real eigen values using $\mathbb{R}_{\exp}$

We saw in section 3.2 that to decide the reachability even in low dimensions even for special matrices, we would need to deal with numbers of the form a^b where both a and b are algebraic. But we saw in the previous subsection that we can express those kind of numbers in $\mathbb{R}_{\exp}$. In this subsection, we will show that assuming decidability of first order theory of $\mathbb{R}_{\exp}$, we can solve for reachability for matrices whose eigen values are all real. Before we prove the main result, we state and prove some useful lemmas and observations.

Consider a matrix A with algebraic entries whose eigenvalues are all real. Let $A = P \Lambda P^{-1}$ be its jordan decomposition where the matrices P, P^{-1}, Λ all have entries which are algebraic. w.l.o.g assume that the jordan blocks in Λ are placed in decreasing order of eigen values in the blocks. We now focus on the $(\Lambda, P^{-1}\mathbb{U})$ system. We know that we can write the matrix Λ as $\begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}$ where Λ_1 has all the non-negative eigen values and Λ_2 has all negative eigen values. If we split P^{-1} accordingly as $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, then we know from theorems Theorem 1 and Lemma 2 that reachable region is of the form $\mathbb{R}^{n_1} \times \mathcal{R}$ where n_1 is the multiplicity of all the non-negative eigen values and $\mathcal{R}$ is the reachable region for the system

$$X' = \Lambda_2 X + B_2 u(t)$$

. Hence from now on, we only consider matrices A which are jordan block diagonal matrices with negative eigen values and matrices B whose number of columns is $\geqslant$ number of rows and $\text{rank}(B) = \#$ of rows.
Call the matrix A to be supernormal if A has all real eigen values and when decomposed into above form we have that (Λ_2, B_2) is normal. Observe that supernormality is a strictly stronger requirement than normality.
If we have $X' = AX + Bu(t)$ with A an $n \times n$ matrix as above and B an $n \times m$ matrix with $u(t) \in [-1, 1]^m$, with (A, B) being normal, then we know that the boundary of the reachable set is given by

$$\partial \mathcal{R} = \left\{ \sum_{i=1}^{m} \int_0^\infty e^{A\tau} b_i \,\text{sgn}(c^T e^{A\tau} b_i)\ d\tau\ | c \in \mathbb{R}^n \setminus \{\mathbf{0}\} \right\}$$

where b_i represents the i^{th} column of B.

Lemma 3. *For a constant vector $b \in \mathbb{R}^n$ and an invertible matrix A one has*

$$\int_0^t e^{As} b\ ds = A^{-1}(e^{At} - I)b$$

. Moreover, if all the eigenvalues of A have real part < 0 then $\lim_{t\to\infty} e^{At} = 0$

Proof. Let $\int_0^t e^{As} b\ ds = F(t)$, we have $F(0) = 0$ and $F'(t) = e^{At} b$. Clearly $g(t) = A^{-1}(e^{At} - I)b$ satisfies the above two conditions and hence by uniqueness of $F(t)$, we have that $\int_0^t e^{As} b\ ds = A^{-1}(e^{At} - I)b$.
$\lim_{t\to\infty} e^{At} = P(\lim_{t\to\infty} e^{Jt})P^{-1}$. e^{Jt} has terms of the form $c.e^{(\lambda_j t)} t^i$ which $\to 0$ if all λ_j have negative real parts. ☐

Lemma 4. *The function $f_{c,A,b}(t) = c^T e^{At} b$ is expressible in $\mathbb{R}_{\exp}$ and we have that $f_{c,A,b}(t)$ has at most $n-1$ zeros.*

The proof for the above lemma can be found in [6]

Theorem 3. *Reachability is decidable for supernormal matrices assuming the first order theory of reals with exponential function* ($\mathbb{R}_{\exp}$) *is decidable.*

Proof. For each algebraic vector $y_0 \in \mathbb{R}^n$, we give a formula η_{y_0} in $\mathbb{R}_{\exp}$ such that y_0 is reachable $\iff$ η_{y_0} is true.
Since $f_{c,A,b}(t)$ has at most $n-1$ zeros, it can have at most $n-1$ sign changes. For each $1 \leqslant i \leqslant m$ and for each $0 \leqslant j \leqslant n-1$, Define two formulas $\xi^{+}_{i,j}(y,c)$ and $\xi^{-}_{i,j}(y,c)$ as following.

$\xi^{+}_{i,j}(y,c) = \exists t_1, t_2, \dots, t_j, 0 < t_1 < t_2 < \dots < t_j \wedge \forall t > 0\,((0 < t < t_1 \implies f_{c,A,b_i}(t) \geqslant 0) \wedge (t_1 < t < t_2 \implies f_{c,A,b_i}(t) \leqslant 0)\dots)$

$\wedge y = \sum_{k=1}^{j} (-1)^{k-1} 2A^{-1}(e^{At_k})b_i - A^{-1}b_i$. Similarly define $\xi^{-}_{i,j}$ this time enforcing the initial sign of $f_{c,A,b_i}(t)$ to be negative. Now define η_{y_0} to be

$$\eta_{y_0} = \exists c, y_1, y_2, \dots, y_m (\in \mathbb{R}^n), \rho(\in \mathbb{R}) \bigwedge_{1 \leqslant i \leqslant m} \left(\bigvee_{0 \leqslant j \leqslant n-1, s \in \{+,-\}} \xi^{s}_{i,j}(c, y_i) \right) \wedge \sum y_i = \rho y_0 \wedge \|c\| = 1 \wedge \rho > 1$$

. By construction, it should be clear that y_0 is reachable $\iff$ η_{y_0} is true. □

3.6 Reachability for 2D matrices with complex eigen values using $\mathbb{R}_{\exp,\cos,\sin}$

Similar to the previous subsection, here, we will solve for reachability in 2 dimensional matrices with complex eigen values assuming decidability of theory of $\mathbb{R}_{\exp,\cos,\sin}$.
In the subsection concerning the 2-dimensional matrices with complex eigen values, we saw that the both the co-ordinates of the boundary points involve some algebraic numbers and also combinations of $e^{\lambda t_1}, e^{\lambda t_2}$ $\cos t_1, \cos t_2$ and $\sin t_1, \sin t_2$ where t_1 and t_2 are the smallest positive reals where certain phase shifted sine functions vanish. Also, we have the term $e^{\frac{\lambda \pi}{\theta}}$ but since we can represent π in our theory ($f(x) = x > 0 \wedge \sin x = 0 \wedge (\forall y 0 < y < x \implies \sin y \neq 0)$) and we have the exponential function, it is not going to be a problem. In the same spirit as the preceeding section, if we could represent t_1, t_2 and the required algebriac numbers in our theory with a formula which uses ϕ and the numbers in P^{-1} as free variables, then it is clear that we can represent the boundary points, which helps us in deciding the reachability problem.
The characterisation of t_1 and t_2 can be formalized in our theory. Recall that t_1 is the smallest real such that $f(s) = \sin(\theta.s + \phi + \beta) = 0$. We use the formula $f(x, \phi, \beta) = x > 0 \wedge \sin(\theta.x + \phi + \beta) = 0 \wedge (\forall y \; 0 < y < x \implies \sin(\theta.y + \phi + \beta) \neq 0)$. Note that it is easy to bound t_1 and t_2 by some constant (say $\max(\lceil \frac{4\pi}{\theta} \rceil, \lceil 4\pi \rceil)$) so that we only need to consider restriced intervals. This proves the required result.

3.7 Time bounded reachability

A variant of the reachability problem is the time bounded version in which we ask if it is possible to reach a certain point in time $\leqslant T$, where T will be part of the input.
Consider the system (1) with the control set $\mathbb{U}$ to be $[-1,1]^n$. Denote by $\mathcal{R}_T(0)$ the set of reachable points from origin in time $\leqslant T$. Then we have the following theorem from [10]

Theorem 4. *If A is normal, then $\mathcal{R}_T(0)$ is closed,bounded,strictly convex. Furthermore, the boundary of the reachable set is given by*

$$\partial \mathcal{R}_T(0) = \left\{ \sum_{i=1}^{n} \int_0^T e^{As} e_i . \operatorname{sgn}(c^T e^{As} e_i)\, ds \;\middle|\; \|c\| = 1 \right\}$$

The condition of A being normal ensures that $c^T e^{As} e_i$ is not identically zero for any c and i. This implies that $c^T e^{As} e_i$ vanishes only finitely many times in the interval $[0,T]$ since the function is analytic. Now, if we could obtain n uniform bounds on the number of zeros for all such functions $c^T e^{As} e_i$ with varying c's, then we can express the boundary points in $\mathbb{R}_{\exp,\cos\restriction_{[0,N]},\sin\restriction_{[0,N]}}$ for a suitable number N.
Observe that each co-ordinate of the vector $e^{As} e_i$ is an exponential polynomial in s *i.e.* it is of the form $\sum_{k=1}^{l} P_k(s) e^{\lambda_k s}$ where P_k may have complex co-efficients and λ_k may be complex. Taking the inner product with a vector c would once again give an exponential polynomial. In [15], Tijdeman showed that for sums of the form $f(z) = \sum_{k=1}^{l} P_k(z) e^{\lambda_k z}$ where P_k is a polynomial of degree $\rho_k - 1$, the number of zeros in the complex plane in any ball of radius R is bounded by $3(n_0 - 1) + 4R\Delta$ where $n_0 = \sum_{k=1}^{l} \rho_k$ and $\Delta = max_k |\lambda_k|$.
For each of the functions in the set $\{c^T e^{As} e_i \mid \|c\| = 1, 1 \leqslant i \leqslant n\}$, it should be clear that the number of zeros is less than $4n + 4R\rho$ where ρ is maximum modulus of an eigenvalue of A. Now, assuming the change of sign for $c^T e^{As} e_i$ occur at $t_1^{c,i}, t_2^{c,i}$ etc., we can symbolically evaluate the integral and therefore could express the boundary point in $\mathbb{R}_{\exp,\cos,\sin}$. Therefore, we showed the following result.

Theorem 5. *When the matrix A is normal, the time bounded reachability problem for (1) is decidable assuming the first order theory of the structure* $\langle \mathbb{R}, +, ., \exp, \cos\restriction_{[0,N]}, \sin\restriction_{[0,N]} \rangle$ *is decidable*

4 Hardness

In this section we try to show that the decidability of the point reachability problem implies the effectiveness of certain Diophantine approximations. This highlights the significant mathematical difficulty in solving this problem.

We follow [3]. Define the *(homogeneous Diophantine approximation) type* of $a \in \mathbb{R}$ to be

$$L(a) = \inf\left\{c : \left|a - \frac{p}{q}\right| < \frac{c}{q^2} \text{for some } p, q \in \mathbb{Z}\right\}.$$

For a given algebriac number a, deciding whether $L(a) > 0$ or not is an open problem. In particular,we do not know a concrete example of an algebraic number of degree > 3 for which $L(a) = 0$ or $L(a) > 0$. Our goal is to prove the following result:

Theorem 6. *If the Point Reachability problem is decidable for bounded convex polytope controls, then for every real positive algebraic number a, $L(a)$ is computable with arbitrary precision.*

One approach we tried is to reduce computability of $L(a)$ to another intermediate reachability problem which doesn't involve any controls and then reduce this problem to point reachability . The thinking was based on the hardness result in [3], in which they use functions which can be seen as exponential polynomials to compute $L(a)$.
In the following proposition, initially we aimed to reduce this another reachability problem without controls to the point reachbility problem . But it was not possible to do it and ultimately we kind of reduced a set reachability version.

Proposition 7. *Let A, B, C be matrices and x_0, y_0 vectors. Let x, y be the solutions to*

$$\begin{aligned} x(0) &= x_0, & x'(t) &= Ax(t) + By(t), \\ y(0) &= y_0, & y'(t) &= Cy(t). \end{aligned} \tag{4}$$

Fix $M > 0$, $\hat{x}$ a vector. Then there exists a (effectively computable) bounded convex polyhedron $\mathbb{U}$, a value ϵ_0, a vector z_0 and a matrix D such that the system

$$\begin{aligned} \tilde{x}(0) &= x_0, & x'(t) &= A\tilde{x}(t) + B\tilde{y}(t), \\ \tilde{y}(0) &= y_0, & y'(t) &= C\tilde{y}(t) + u(t), \\ \tilde{z}(0) &= 0, & z'(t) &= D\tilde{z}(t) + v(t), \end{aligned} \tag{5}$$

has the following properties for all $0 < \varepsilon < \epsilon_0$

- *if there exists $T \geqslant 0$ such that $x(T) = \hat{x}$ and $\|y(T)\| \leqslant M$, then there exists $(u, v) : [0, +\infty) \to \mathbb{U}$ measurable and $T' \geqslant 0$ such that $\|\tilde{x}(T') - \hat{x}\| \leqslant \varepsilon$, $\tilde{y}(T') = 0$ and $\tilde{z}(T') = z_0$,*
- *if there exists $(u, v) : [0, +\infty) \to \mathbb{U}$ measurable and $T \geqslant 0$ such that $\|\tilde{x}(T) - \hat{x}\| \leqslant \varepsilon$, $\tilde{y}(T) = 0$ and $\tilde{z}(T) = z_0$, them there exists $T' \geqslant 0$ such that $\|x(T') - \hat{x}\| \leqslant 4\varepsilon$ and $\|y(T')\| \leqslant 6M$.*

Proof. Define

$$\delta = \min\left(\|A\|^{-1}, \|C\|^{-1}, \varepsilon(2\|A\|\|\hat{x}\| + 6\|B\|M)^{-1}\right) \quad \text{and} \quad \epsilon_0 = \|\hat{x}\| + 3\|B\|M\|A\|^{-1}.$$

Let $z_0 = \begin{bmatrix} \delta \\ \frac{1}{2}\delta^2 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Define $K = 3M\delta^{-1}$ and $\mathbb{U} = \{(u, v_1, 0) : \|u\| \leqslant K|v_1|, 0 \leqslant v_1 \leqslant 1\}$. Note that $\mathbb{U}$ is a bounded convex polyhedron containing 0. From now on, we use $v(t)$ to describe $v_1(t)$ as the 2^{nd} coordinate is always 0. Recall that, given v, $\tilde{z}$ is the solution to $\tilde{z}(0) = 0$ and $\tilde{z}'(t) = D\tilde{z}(t) + v(t)$.

Claim 2. *Let $T > 0$, then $\tilde{z}(T + \delta) = z_0 \iff v(t) = U_T(t)$ almost everywhere, where $U_T : [0, T + \delta] \to [0, 1]$ is defined by $U_T(t) = \begin{cases} 0 & \text{if } t \leqslant T \\ 1 & \text{if } T < t \leqslant T + \delta \end{cases}$.*

We defer the proof of the claim and now prove the two statements. For the first statement, let $T' = T + \delta$. Then by the above claim we have $\tilde{z}(T') = z_0$ as soon as that $v(t) = U_T(t)$. From the definition of $\mathbb{U}$, we must take $u(t) = 0$ for $t \in [0, T]$ and can choose any $u(t) \in [-K, K]^n$ for $T < t \leqslant T + \delta$. We now show how to build a control that steers $\tilde{y}$ to 0 at time T'. Let $u(\tau) = -e^{C(\tau - T)}y(T)\delta^{-1}$ for $T < \tau \leqslant T + \delta$. Observe that

$$\|u(\tau)\| = \|e^{C(\tau-T)}y(T)\delta^{-1}\| \leqslant \|e^{C(\tau-T)}\|\|y(T)\|\delta^{-1} \leqslant e^{\|C\|(\tau-T)}M\delta^{-1} \leqslant e^{\|C\|\delta}M\delta^{-1} \leqslant eM\delta^{-1} \leqslant K,$$

Therefore $u(\tau)$ is a valid control. Also, since $u(t)$ is 0 for $t \leqslant T$ we have that $\tilde{y}(t) = y(t)$ and therefore $\tilde{x}(t) = x(t)$ for all $t \leqslant T$. In particular, we have that $\tilde{y}(T) = y(T)$ and $\tilde{x}(T) = x(T) = \hat{x}$. From the ODE of $\tilde{y}$ we obtain that

$$\tilde{y}(t) = e^{C(t-T)}y(T) + \int_T^t e^{C(t-s)}u(s)\, ds$$

and substituting the above control gives

$$\tilde{y}(t) = e^{C(t-T)}y(T) - \int_T^t e^{C(t-s)}e^{C(s-T)}y(T)\delta^{-1}\, ds = e^{C(t-T)}y(T)\Big(1-(t-T)\delta^{-1}\Big)$$

for $T < t \leqslant T'$. In particular, $\tilde{y}(T') = 0$ and

$$\|\tilde{y}(t)\| \leqslant e^{\|C\|(t-T)}\|y(T)\| \leqslant eM \leqslant 3M$$

for all $T < t \leqslant T'$ since $t - T \leqslant \delta$ and $\|C\|\delta \leqslant 1$. Similarly we have

$$\tilde{x}(t) = e^{A(t-T)}\hat{x} + \int_T^t e^{A(t-s)}By(s)\, ds \quad \text{and thus} \quad \tilde{x}(T') = e^{A\delta}\hat{x} + \int_T^{T'} e^{A(T'-s)}By(s)\, ds.$$

It follows that

$$\|\tilde{x}(T') - \hat{x}\| = \|(e^{A\delta} - I)\hat{x} + \int_T^{T'} e^{A(T'-s)}By(s)\, ds\| \leqslant \|(e^{A\delta} - I)\hat{x}\| + \|\int_T^{T'} e^{A(T'-s)}By(s)\, ds\|$$

but

$$\begin{aligned}
\|\int_T^{T'} e^{A(T'-s)}By(s)\, ds\| &\leqslant \int_T^{T'} \|e^{A(T'-s)}By(s)\, ds\| \\
&\leqslant \int_T^{T'} \|e^{A(T'-s)}\|\|B\|\|y(s)\|\, ds \\
&\leqslant \int_T^{T'} e^{\|A\|(T'-s)}\|B\|3M\, ds \\
&= 3M\|B\|(e^{\|A\|\delta} - 1)\|A\|^{-1} \\
&\leqslant 6M\|B\|\delta \qquad\qquad \text{since } e^{\alpha} - 1 \leqslant 2\alpha \text{ for } \alpha = \|A\|\delta \leqslant 1.
\end{aligned}$$

Similarly, $\|e^{A\delta} - I\| \leqslant 2\|A\|\delta$ since $\|A\|\delta \leqslant 1$ and therefore $\|\tilde{x}(T') - \hat{x}\| \leqslant (2\|A\|\hat{x} + 6\|B\|M)\delta \leqslant \varepsilon$ which is what we wanted.

For the second statement, since $\tilde{z}(T) = z_0$ then by the above claim, $T \geqslant \delta$ and $v(t) = U_{T'}(t)$ where we let $T' = T - \delta$. The definition of $\mathbb{U}$ and $U_{T'}$ implies that $u(t) = 0$ for $t \leqslant T'$, therefore $\tilde{x}(t) = x(t)$ and $\tilde{y}(t) = y(t)$ for $t \leqslant T'$ and $u(t) \in [-K, K]^n$ for $t \in (T', T]$. From the ODE of $\tilde{y}$ and using $\tilde{y}(T) = 0$, we have that

$$\tilde{y}(t) = e^{C(t-T)}\tilde{y}(T) + \int_T^t e^{C(t-s)}u(s)\, ds = -\int_t^T e^{C(t-s)}u(s)\, ds$$

for $T' < t \leqslant T$. Thus

$$\begin{aligned}
\|\tilde{y}(t)\| &\leqslant \int_t^T e^{\|C\||(t-s)|}\|u(s)\|\, ds \\
&\leqslant 3M\delta^{-1}\int_t^T e^{\|C\|(s-t)}\, ds \\
&= 3M\delta^{-1}(e^{\|C\|(T-t)} - 1)\|C\|^{-1} \\
&\leqslant 3M\delta^{-1}(e^{\|C\|\delta} - 1)\|C\|^{-1} \\
&\leqslant 3M\delta^{-1}2\delta = 6M.
\end{aligned}$$

In particular $\|\tilde{y}(T')\| = \|y(T')\| \leqslant 6M$. Similarly, from the ODE of $\tilde{x}$ we get that

$$\tilde{x}(t) = e^{A(t-T)}\tilde{x}(T) + \int_T^t e^{A(t-s)}B\tilde{y}(s)\, ds$$

Substituting T' for t we get in particular that

$$\begin{aligned}
\|\tilde{x}(T') - \tilde{x}(T)\| &\leqslant \|e^{-A\delta} - I\|\|\tilde{x}(T)\| + \int_{T'}^{T} e^{\|A\||T'-s|}\|B\|\|\tilde{y}(s)\|\, ds \\
&\leqslant 2\|A\|\delta\|\tilde{x}(T)\| + 6M\|B\|\int_{T'}^{T} e^{\|A\|(s-T')}\, ds \\
&\leqslant 2\|A\|\delta(\|\hat{x}\| + \varepsilon) + 6M\|B\|(e^{\|A\|\delta} - 1)\|A\|^{-1} && \text{since } \|\tilde{x}(T) - \hat{x}\| \leqslant \varepsilon \\
&\leqslant 2\|A\|\delta(\|\hat{x}\| + \varepsilon) + 12M\|B\|\delta \\
&\leqslant (2\|A\|\|\hat{x}\| + 12M\|B\|)\delta + 2\|A\|\delta\varepsilon \\
&\leqslant (4\|A\|\|\hat{x}\| + 12M\|B\|)\delta + \varepsilon^2(\|\hat{x}\| + 3\|B\|M\|A\|^{-1})^{-1} && \text{by definition of } \delta \\
&\leqslant 2\varepsilon + \varepsilon^2\epsilon_0^{-1} \\
&\leqslant 3\varepsilon && \text{since } \varepsilon < \epsilon_0.
\end{aligned}$$

It follows that $\|x(T') - \hat{x}\| \leqslant \|x(T') - \tilde{x}(T)\| + \|\tilde{x}(T) - \hat{x}\| \leqslant 4\varepsilon$.

Proof of Claim 2. Let us start by writing the statement in a more direct way.

$$\text{Given } z_0 = \begin{bmatrix} \delta \\ \frac{1}{2}\delta^2 \end{bmatrix}, \begin{bmatrix} z_1' \\ z_2' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix} \text{ where } 0 \leqslant v(t) \leqslant 1 \text{ and } z_1(0) = z_2(0) = 0. \text{ Then}$$

$$\exists T \geqslant 0 \text{ such that } z(T+\delta) = z_0 \iff v(t) = U_T(t) \text{ almost everywhere, where}$$

$$U_T : [0, T+\delta] \to [0,1] \text{ is defined by } U_T(t) = \begin{cases} 0 & \text{if } t \leqslant T \\ 1 & \text{if } T < t \leqslant T+\delta \end{cases}.$$

We will first prove the reverse implication which is simpler and direct to prove.
$(\Longleftarrow)$
First, let us deal with the 1^{st} co-ordinate. We have $z_1' = v(t)$ which is equal to $U_T(t)$ by hypothesis.
$\implies z_1'(t) = \begin{cases} 0 & \text{if } t \leqslant T \\ 1 & \text{if } T < t \leqslant T+\delta \end{cases}$. This along with the initial condition that $z_1(0) = 0$ implies that
$z_1(t) = \begin{cases} 0 & \text{if } t \leqslant T \\ t - T & \text{if } T < t \leqslant T+\delta \end{cases}$. In particular we have that $z_1(T+\delta) = \delta$. Now, for the 2^{nd} co-ordinate

$$z_2'(t) = z_1(t) \implies \int_0^t z_2'(s)\, ds = \int_0^t z_1(s)\, ds \implies z_2(t) = \begin{cases} 0 & \text{if } t \leqslant T \\ \frac{1}{2}(t-T)^2 & \text{if } T < t \leqslant T+\delta \end{cases}$$

$\implies z_2(T+\delta) = \frac{1}{2}\delta^2 \implies z(T+\delta) = z_0$
$(\Longrightarrow)$
Now assume that there is a control $v(t)$ such that $z(T+\delta) = z_0$.
We would like to prove that $v(t) = U_T(t)$ almost everywhere. Let $D = \{t : v(t) \neq U_T(t)\ 0 \leqslant t \leqslant T+\delta\}$ be the set of times where the both controls differ and assume that $|D| > 0$ where $|D|$ denotes the measure of the set.
Observe that $\forall\, t \in [T, T+\delta]$ we must have that $z_1(t) \geqslant t - T$, as $z_1(t) < t - T \implies$

$$\int_t^{T+\delta} z_1'(s)\, ds \leqslant \int_t^{T+\delta} ds \text{ since } z_1' = v \leqslant 1$$

$$\implies z_1(T+\delta) \leqslant T + \delta - t + z_1(t) < T + \delta - t + t - T = \delta \text{ a contradiction}$$

Also, since $v(t) \geqslant 0$ and $z_1(t) = \int_0^t v(s)\, ds$, we have that $z_1(t) \geqslant 0$. Combining both, we get that $z_1(t) \geqslant \max(0, t-T)\ \forall\ 0 \leqslant t \leqslant T+\delta$.
$|D| = |\{t : v(t) \neq U_T(t)\ 0 \leqslant t \leqslant T+\delta\}| = |\{t : v(t) \neq 0\ ,\ 0 \leqslant t \leqslant T\}| + |\{t : v(t) \neq 1\ ,\ T \leqslant t \leqslant T+\delta\}|$
Name the first measure $D_{[0,T]}$ and the second $D_{[T,T+\delta]}$.
$\because |D| > 0$, this would imply at least one of $D_{[0,T]}$, $D_{[T,T+\delta]}$ to be > 0. Assume $D_{[0,T]} = 0$. We have

$$z_1(T+\delta) = \int_0^{T+\delta} v(s)\, ds = \delta = \int_0^{T+\delta} U_T(s)\, ds$$

$$\implies \int_0^{T+\delta} v(s) - U_T(s)\, ds = \int_0^T v(s)\, ds + \int_T^{T+\delta} v(s) - 1\, ds = 0$$

But

$$D_{[0,T]} = 0 \implies \int_0^T v(s)\, ds = 0 \text{ and } D_{[T,T+\delta]} > 0 \implies \int_T^{T+\delta} v(s) - 1\, ds < 0$$

And since the sum of 0 and a negative number cannot be 0 we must have $D_{[0,T]} > 0$ which implies

$$z_1(T) = \int_0^T v(s)\, ds > 0$$

But now,

$$\begin{aligned} z_2(T+\delta) &= \int_0^T z_1(s)\, ds + \int_T^{T+\delta} z_1(s)\, ds \\ &\geqslant \int_0^T z_1(s)\, ds + \int_T^{T+\delta} s - T\, ds \\ &= \int_0^T z_1(s)\, ds + \tfrac{1}{2}\delta^2 \\ &> \tfrac{1}{2}\delta^2 & \because z_1(T) > 0 \text{ and } z_1 \text{ is continuous} \end{aligned}$$

which leads to a contradiction. $\therefore |D| = 0$ and $v(t) = U_T(t)$ almost everywhere in $[0, T+\delta]$ □

□

Proposition 8. *Let a be a positive real algebraic number, $c, \varepsilon > 0$. Then there exists (effectively computable) matrices A, B, C and vectors x_0, y_0 such that the solution x, y to*

$$\begin{aligned} x(0) &= x_0, & x'(t) &= Ax(t) + By(t), \\ y(0) &= y_0, & y'(t) &= Cy(t) \end{aligned} \tag{6}$$

satisfies that

- *if $L(a) \leqslant c$ then there exists $T > 0$ such that $x(T) = 0$ and $\|y(T)\| \leqslant f(c, a)$,*
- *if there exists $T > 0$ such that $\|x(T)\| \leqslant \varepsilon$ then $L(a) \leqslant g(c)$.*

We didn't tryout the above proposition as the one before was changed considerably. Instead we tried to directly reduce computability of $L(a)$ to the point reachability problem.

4.1 Some results regarding diophantine approximations

In this subsection, we show that the computability of $L(a)$ to arbitrary precision is equivalent to checking if some certain functions are negative at some periodic points.

Proposition 9. *Let a be a positive real algebraic number and $0 < c < 1$. Define $f(t) = t|\sin at| - 4\pi^2 c$ and $g(t) = t^2(1 - \cos at) - 8\pi^4 c^2$. Then*

- *If $L(a) \leqslant c$ is witnessed by some p, q with $q > 20$, then $f(t) < 0$ for some $t = 2\pi K$, $K \in \mathbb{N}$*
- *If $L(a) \leqslant c$ is witnessed by some p, q with $q > 20$, then $g(t) < 0$ for some $t = 2\pi K$, $K \in \mathbb{N}$*
- *If $f(t) < 0$ for some $t = 2\pi q$, $q \in \mathbb{N}$ and $q > 20$, then $L(a) \leqslant 8c$*
- *If $g(t) < 0$ for some $t = 2\pi q$, $q \in \mathbb{N}$ and $q > 20$, then $L(a) \leqslant 1.5c$*

Proof.

- Assume $L(a) \leqslant c \implies \exists\, p,q$ such that $\left|a - \frac{p}{q}\right| < \frac{c}{q^2}$. Define $t = 2\pi q$

  $\implies |aq - p| < \frac{c}{q} \implies |a * 2\pi q - 2\pi p| < \frac{2\pi c}{q} \implies |at - 2\pi p| < \frac{2\pi c}{q}$

  $$|\sin at| = |\sin(at - 2\pi p)| = \sin(|at - 2\pi p|) \quad \because |at - 2\pi p| < \frac{2\pi c}{q} < 1$$

  Now using $\sin x \leqslant x$ for $x > 0$, we get

  $$\implies t|\sin at| \leqslant t|at - 2\pi p| < \frac{2\pi c}{q} * 2\pi q = 4\pi^2 c$$

  which implies that $f(t) < 0$ for $t = 2\pi q$

- Similarly, using $1-\cos x < \frac{x^2}{2}$ and $\cos x = \cos|x|$, for $t = 2\pi q$, we get

$$t^2(1-\cos at) = t^2(1-\cos(|at-2\pi p|)) < 4\pi^2 q^2 * \frac{|at-2\pi p|^2}{2} < 2\pi^2 q^2 * \frac{4\pi^2 c^2}{q^2} = 8\pi^4 c^2$$

$\implies g(t) < 0$ for $t = 2\pi q$

- Given $f(t) < 0$ for some $t = 2\pi q$. $\implies |\sin at| < 4\pi^2 c/t = 2\pi c/q$.
Let $p \in \mathbb{N}$ such that $2\pi qa = 2\pi p + \delta$ with $-\pi < \delta \leqslant \pi$. Then $|\sin at| = \sin(|at-2\pi p|) = \sin|\delta|$
$\implies$ we have $\sin|\delta| < 2\pi c/q < 1/2$ which would mean that either $|\delta| \in [0, \pi/6]$ or $|\delta| \in [\pi - \pi/6, \pi]$
If $|\delta| \in [0, \pi/6]$ then $|\delta|/2 < \sin|\delta| < 2\pi c/q$ which gives $|a - \frac{p}{q}| < \frac{2c}{q^2} \implies L(a) \leqslant 2c$
On the other hand if $|\delta| \in [\pi - \pi/6, \pi]$ then we will have $\frac{\pi - |\delta|}{2} < sin|\delta| < 2\pi c/q$

$$\implies \pi - |\delta| < \frac{4\pi c}{q}$$

Let $t' = 2t$. Then $at' = 2\pi a(2q) = 2\pi(2p) + 2\delta = 2\pi p' + \delta'$ where $\delta' = 2\delta \pm 2\pi$ and $p' = 2p \mp 1$ respectively depending on the sign of δ. Note that δ' and δ will necessarily have different signs and $|\delta'| = 2(\pi - |\delta|)$

$$\implies |\delta'| = |at' - 2\pi p'| < \frac{8\pi c}{q} \implies |a - \frac{p'}{2q}| < \frac{8c}{(2q)^2}$$

which implies $L(a) \leqslant 8c$

- Given $g(t) < 0$ for some $t = 2\pi q$. $\implies \cos at > 1 - \frac{8\pi^4 c^2}{4\pi^2 q^2} = 1 - 2\pi^2 c^2 q^{-2} > 3/4 > 0$.
Let $p \in \mathbb{N}$ such that $2\pi qa = 2\pi p + \delta$ with $-\frac{\pi}{2} < \delta \leqslant \frac{\pi}{2}$. Then $\cos at = \cos(at - 2\pi p) = \cos\delta = \cos|\delta|$
when $0 < x < \pi/2$ we have $1 - \frac{2}{9}x^2 > \cos x$
$\implies 1 - \frac{2}{9}|\delta|^2 > \cos|\delta| > 1 - \frac{2\pi^2 c^2}{q^2} \implies |\delta|^2 < \frac{9\pi^2 c^2}{q^2} \implies 2\pi q|a - \frac{p}{q}| < \frac{3\pi c}{q}$
$\implies L(a) \leqslant 1.5c$

□

We would like to construct an instance of point reachability problem such that the answer to that instance is yes when $f(t)$ (or $g(t)$) is < 0 and vice versa. But unfortunately,we weren't able to construct such an instance. An instance of our problem means to find a matrix A, and a bounded control set $\mathbb{U}$ and two vectors x_1 and x_2 serving as initial and final points. We tried two matrices A, for both of which we couldn't find a suitable control set either bounded or even unbounded.

4.2 Positivity hardness of set reachability

In this subsection we introduce the positivity problem for continuous time systems and show that set reachability when the target and control sets are bounded convex polytopes is positivity hard.

Problem (Continuous Positivity Problem). *Given a real matrix $A \in \mathbb{A}^{n\times n}$ and two real vectors c , $x_0 \in \mathbb{A}^n$, $\exists t > 0$ such that $c^T \exp(At)x_0 < 0$?*

Theorem 10. *Set reachability is positivity hard*

Proof. We reduce continuous positivity to set reachability. Given c and $x_0 \in \mathbb{A}^n$ and matrix $A \in \mathbb{A}^{n\times n}$.
Define $f(t) = c^T \exp(At)x_0$. Observe that $f(t)$ is an exponential polynomial *i.e.* $f(t)$ can be expressed as
$\Sigma_{j=1}^m e^{\lambda_j t}(P_{1j}(t)\cos(\theta_j t) + P_{2j}(t)\sin(\theta_j t))$ where P_{1j} and P_{2j} are polynomials and some of the $\theta_j' s$ maybe zero.

Define $\alpha = -\max_i(|\lambda_i|) - 1$ and $g(t) = e^{\alpha t} f(t)$. Observe that $g(t) \to 0$ as $t \to \infty$. Since $g(t)$ is an exponential polynomial, we can compute a matrix B of appropriate order and a vector y_0 such that the 1^{st} co-ordinate of the solution of initial value problem given by

$$y' = By \quad , \quad y(0) = y_0$$

is equal to $g(t)$ and every other co-ordinate in the vector $y(t)$ is bounded by an effective computable constant M.

Assuming, for now, the existence and computability of such matrix B and the vector y_0, the instance of the set reachability we consider is pretty straightforward. Consider the system

$$y' = By + u(t) \quad , \quad y(0) = y_0 \text{ where } u(t) \in \mathbb{U} = \{0\} \text{ and } \mathbb{T} = [-1, 0] \times [-M, M]^{(n-1)} \tag{7}$$

Clearly, if the answer to (7) is true, then we have a yes instance of the positivity problem. Also if we have $f(t) < 0$ for some t, then either $g(t) \in [-1, 0]$ and we are done or if $g(t) < -1$ then by continuity of $g(t)$ and knowing that $g(t) \to 0$ as $t \to \infty$, using Intermediate value theorem we can be sure of an existence of a t' such that $g(t') \in [-1, 0]$ which completes the proof. We needn't worry about other co-ordinates of y as they are always in $[-M, M]$

To construct B, we work backwards. We have $g(t) = \Sigma_{j=1}^{m} e^{\lambda'_j t}(P_{1j}(t)\cos(\theta_j t) + P_{2j}(t)\sin(\theta_j t))\ \lambda'_j < 0\ \forall j$. For each j, let n_j denote $\max(\deg(P_{1j}), \deg(P_{2j}))$. Define auxiliary variables x_{1i}^j and x_{2i}^j for $0 \leqslant i \leqslant n_j$ where $x_{1i}^j(t) = e^{\lambda'_j t} t^i \cos\theta_j t$ and $x_{2i}^j(t) = e^{\lambda'_j t} t^i \sin\theta_j t$. Let $\mathbf{x_j} = \begin{bmatrix} x_{10}^j \cdots x_{2n_j}^j \end{bmatrix}^T$. Since derivative of each of the co-ordinates in $\mathbf{x_j}$ can be expressed as linear combination of co-ordinates in $\mathbf{x_j}$, we can compute a matrix $\mathbf{A_j}$ such that $\mathbf{x'_j} = \mathbf{A_j x_j}$. Now $g(t)$ can be written as $\Sigma_{j=1}^{m} \mathbf{d_j^T x_j}$ for some vectors $\mathbf{d_1} \cdots \mathbf{d_m}$ in $\mathbb{R}^n$.

$$B = \begin{bmatrix} 0 & \mathbf{d_1} \cdots \mathbf{d_m} \\ \mathbf{0} & \operatorname{diag}(A_1, A_2, \cdots, A_m) \end{bmatrix} \text{ and } y = \begin{bmatrix} g(t) \\ \mathbf{x_1} \\ \vdots \\ \mathbf{x_m} \end{bmatrix}$$

satisfies the requirements with initial vector chosen accordingly. □

In the above theorem, we can consider the target set to be a scaled hypercube instead of a general convex polytope and the proof still works. It tells us that even for a simple target set such as a hypercube, the set reachability problem is already harder.
Of course, for the above theorem to demonstrate the hardness of set reachability, we need to know how hard the positivity problem is? As far as I know, there is only one result concerning the positivity hardness given by [1] which states that it is atleast NP-Hard.

4.3 Hardness: a gadget

Consider the system

$$\begin{aligned} x(0) &= 0, & x'(t) &= u(t), \\ y(0) &= 0, & y'(t) &= x(t) - 1, \end{aligned} \tag{8}$$

where $u(t) \in \{0, 1\}$ and with target $\mathcal{T} = (1, -\frac{1}{2})$. One readily checks that $x(t) = \int_0^t u(s)ds$ and by integration by part,

$$y(t) = -t + \int_0^t x(s)ds = -t + \big[tx(t)\big]_0^t - \int_0^t sx'(s)ds = t(x(t) - 1) - \int_0^t su(s)ds.$$

But now observe that if $(x(T), y(T)) = \mathcal{T}$ for some $T \geqslant 0$, then $x(T) = 1$, therefore $y(T) = -\int_0^t su(s)ds = -\frac{1}{2}$. Define $E = \{t \in [0, 1] : u(t) = 1\}$ and $F = \{t \in [1, T] : u(t) = 1\}$. Since $u(t) \in \{0, 1\}$, we have that $x(T) = \lambda(E) + \lambda(F) = 1$ where λ denotes the Lebesgue measure. But now observe that

$$\begin{aligned} y(T) &= \int_E sds + \int_F sds \\ &\geqslant \int_0^{\lambda(E)} sds + \int_1^{1+\lambda(F)} sds \\ &= \frac{1}{2}\left(\lambda(E)^2 + (1 + \lambda(F))^2 - 1\right) \\ &= \frac{1}{2}\left(\lambda(E)^2 + \lambda(F)^2 + 2\lambda(F)\right) \\ &= \frac{1}{2}\left((1 - \lambda(F))^2 + \lambda(F)^2 + 2\lambda(F)\right) && \text{since } \lambda(E) + \lambda(F) = 1 \\ &= \frac{1}{2}\left(1 + 2\lambda(F)^2\right) && \text{since } \lambda(E) + \lambda(F) = 1. \end{aligned}$$

But since $y(T) = \frac{1}{2}$, we have that $2\lambda(F)^2 \leqslant 0$, ie $\lambda(F) = 0$ and $\lambda(E) = 1$.

Conversely, check that if $u(t) = 1$ for $t \leqslant 1$ and $u(t) = 0$ elsewhere, then clearly $x(T) = 1$ for $T \geqslant 1$ and $Y(T) = -\int_0^1 sds = -\frac{1}{2}$ so $(x(T), y(T)) = \mathcal{T}$.

Lemma 5. *If u satisfies* (8)*, then $u = \mathfrak{u}$ almost everywhere, where* $\mathfrak{u}(t) = \begin{cases} 1 & \text{if } t \leqslant 1 \\ 0 & \text{elsewhere} \end{cases}$.

4.4 Hardness: a new hope

Let $\varphi = \frac{1+\sqrt{5}}{2}$ be the golden ratio, it is an irrational number that satisfies $\varphi^2 - \varphi - 1 = 0$. Let $(F_n)_n$ be the Fibonacci sequence, defined by

$$F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n.$$

Lemma 6 (Textbook result)**.** *For all $q \geqslant 1$ and $p \in \mathbb{Z}$,*

$$\left|\varphi - \frac{p}{q}\right| > \frac{1}{\sqrt{5}q^2 + \frac{1}{2}q}.$$

On the other hand,

$$\lim_{n\to\infty} F_{n-1}^2 \left|\varphi - \frac{F_n}{F_{n-1}}\right| = \frac{1}{\sqrt{5}}.$$

Question: can we ensure that $\sin(2n\pi\varphi) \approx \frac{1}{\sqrt{5}n}$ for most values of n ? Or maybe choose a different number than $\sqrt{5}$. Quadratic irrationals have well-understood continued fraction expansions and therefore Diophantine types.

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